Practices before the class (March 22)

- (**T/F**) Let A be an $m \times n$ matrix. If the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution, then $\dim(\mathrm{Nul}(A)) = 0$.

- **(T/F)** Eigenvalues must be nonzero scalars.

Practices before the class (March 22)

- (**T/F**) Let A be an $m \times n$ matrix. If the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution, then $\dim(\text{Nul}(A)) = 0$.
  True.
  $A\mathbf{x} = \mathbf{b}$ has a unique solution implies there are no free variables.
  Thus every column is a pivot column. Then $\operatorname{rank} A = n$.
  By the Rank Theorem: $\operatorname{rank} A + \dim(\text{Nul}(A)) =$ number of columns of A, we know $\dim(\text{Nul}(A)) = 0$.

- **(T/F)** Eigenvalues must be nonzero scalars.
  False. Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
  The characteristic equation is $|A - \lambda I| = \lambda(\lambda - 1) = 0$. Thus A has an eigenvalue 0.

5.4 Eigenvectors and Linear Transformations

Eigenvectors of Linear Transformations

Definition. Let V be a vector space. An **eigenvector** of a linear transformation $T : V \to V$ is a nonzero vector $\mathbf{x}$ in V such that $T(\mathbf{x}) = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of T if there is a nontrivial solution $\mathbf{x}$ of $T(\mathbf{x}) = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an **eigenvector** corresponding to λ.

The Matrix of a Linear Transformation

Example 0. Coordinate Vector of a Vector Respect to a Basis

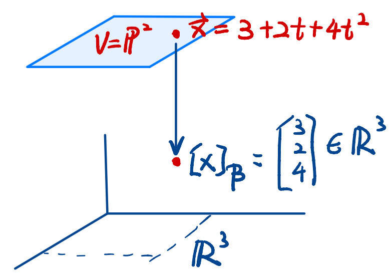
Let V be an n-dimensional vector space with a basis $\mathcal{B}$. Then any $\mathbf{x}$ in V can be viewed as an element in $\mathbb{R}^n$. For example,

- $V = \mathbb{P}_2$, which is the vector space of the polynomials of degree at most 2.

  The standard basis $\mathcal{B} = \{1, t, t^2\}$ and $\dim\{\mathbb{P}_2\} = 3$.

  Consider a vector $\mathbf{x} \in \mathbb{P}_2$, say $\mathbf{x} = 3 + 2t + 4t^2$, then the coordinate vector with respect to $\mathcal{B}$ (recall in § is

  $$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} \in \mathbb{R}^3$$

  Similarly, if $\mathbf{y} = 5 + 3t$, then $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix} \in \mathbb{R}^3$.

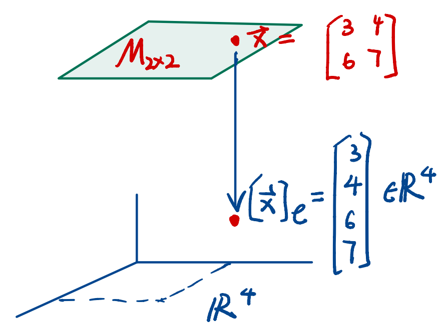
  This can be generalized to the case $\mathbb{P}_n$, i.e., any element in $\mathbb{P}_n$ can be presented as a vector in $\mathbb{R}^{n+1}$ after we choose a basis for $\mathbb{P}_n$.

- $W = \mathbb{M}_{2\times 2}$, which is the vector space of all 2×2 matrices.

  The standard basis $\mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ and $\dim \mathbb{M}_{2\times 2} = 4$.

  Consider a vector $\mathbf{x} = \begin{bmatrix} 3 & 4 \\ 6 & 7 \end{bmatrix} \in \mathbb{M}_{2\times 2}$, then the coordinate vector with respect to $\mathcal{C}$ is

  $$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 4 \\ 6 \\ 7 \end{bmatrix} \in \mathbb{R}^4$$

Now we consider a linear transformation $T : \mathbb{P}_2 \to \mathbb{M}_{2\times 2}$ defined by

$$T(\mathbf{p}(t)) = \begin{bmatrix} \mathbf{p}(0) & \mathbf{p}(2) \\ \mathbf{p}(1) & \mathbf{p}'(2) \end{bmatrix}$$

Can we think about such T in terms of some matrix? (See **Example 1**.)

Recall from Section 1.9 that any linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be implemented via left-multiplication by a matrix A, called the standard matrix of T. We generalize this notion to any linear transformation between two finite-dimensional vector spaces.

Let

- V : an n-dimensional vector space
- W: an m-dimensional vector space
- T : any linear transformation from V to W.

To associate a matrix with T, choose (ordered) bases $\mathcal{B}$ and $\mathcal{C}$ for V and W, respectively.

Given any $\mathbf{x}$ in V, the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ is in $\mathbb{R}^n$ and the coordinate vector of its image, $[T(\mathbf{x})]_{\mathcal{C}}$, is in $\mathbb{R}^m$, as shown in Figure 1.

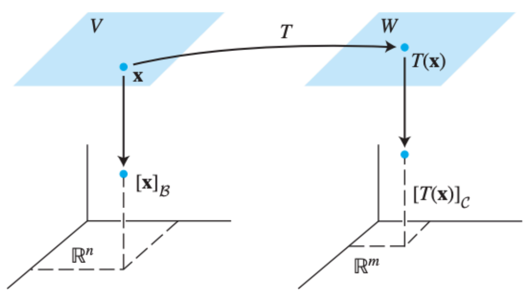


FIGURE 1 A linear transformation from V to W.

We want to find a matrix M such that the action of T on $\mathbf{x}$ may be viewed as left-multiplication by M. See Figure 2.

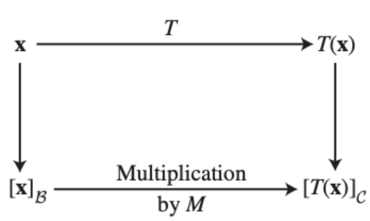


FIGURE 2

We use the following example to show how to find such a matrix M. The general idea is explained on the next page.

Example 1. Define $T : \mathbb{P}_2 \to \mathbb{M}_{2\times 2}$ by $T(\mathbf{p}(t)) = \begin{bmatrix} \mathbf{p}(0) & \mathbf{p}(2) \\ \mathbf{p}(1) & \mathbf{p}'(2) \end{bmatrix}$.

a. Find the image under T of $\mathbf{p}(t) = 5 + 3t$.
b. Show that T is a linear transformation.
c. Find the matrix M for T relative to the basis $\{1, t, t^2\}$ for $\mathbb{P}_2$ and the standard basis for $\mathbb{M}_{2\times 2}$.

ANS: (a). Note $p'(t) = 3$ if $p(t) = 5+3t$.

Thus $T(5+3t) = \begin{bmatrix} 5+3\cdot 0 & 5+3\cdot 2 \\ 5+3\cdot 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ 8 & 3 \end{bmatrix}$

(b). By the definition of a linear transformation, we need to show ① $T(p_1+p_2) = T(p_1) + T(p_2)$
② $T(cp) = cT(p)$, where $p_1, p_2 \in \mathbb{P}_2$ and $c \in \mathbb{R}$.

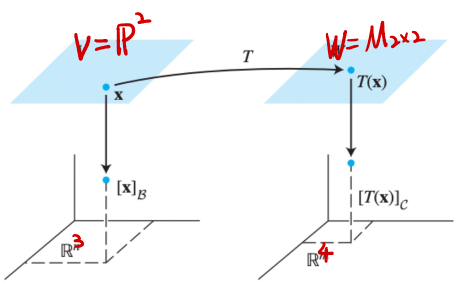
For ① $T(p_1+p_2) = \begin{bmatrix} (p_1+p_2)(0) & (p_1+p_2)(2) \\ (p_1+p_2)(1) & (p_1+p_2)'(2) \end{bmatrix}$

by the property of polynomial and derivatives. $\begin{bmatrix} p_1(0) & p_1(2) \\ p_1(1) & p_1'(2) \end{bmatrix} + \begin{bmatrix} p_2(0) & p_2(2) \\ p_2(1) & p_2'(2) \end{bmatrix}$

$= T(p_1) + T(p_2)$

Similary, you can show ② is true for the given T.

(c).



$$\begin{tikzcd} \mathbf{x} \arrow[r, "T"] \arrow[d] & T(\mathbf{x}) \arrow[d] \\ {[\mathbf{x}]_{\mathcal{B}}} \arrow[r, "\text{Multiplication}", "\text{by } M"'] & {[T(\mathbf{x})]_{\mathcal{C}}} \end{tikzcd}$$

3-dim'l 4-dim'l.

Let $\mathcal{B} = \{1, t, t^2\}$ be a basis for $\mathbb{P}_2$

$\mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ be a basis for $M_{2\times 2}$.

Then the desired matrix is a 4×3 matrix constructed by computing the images of basis elements in $\mathcal{B}$ in terms of the basis $\mathcal{C}$, i.e.

$$\left[[T(1)]_{\mathcal{C}} \quad [T(t)]_{\mathcal{C}} \quad T[(t^2)]_{\mathcal{C}} \right]$$

We compute.

$$T(1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus $[T(1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

$$T(t) = \begin{bmatrix} p(0) & p(2) \\ p(1) & p'(2) \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

Thus

$$[T(t)]_e = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$T(t^2) = \begin{bmatrix} 0 & 2^2 \\ 1^2 & 2\cdot 2 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 1 & 4 \end{bmatrix}$$

Thus

$$[T(t^2)]_e = \begin{bmatrix} 0 \\ 4 \\ 1 \\ 4 \end{bmatrix}$$

Therefore, the matrix for T relative to the basis $\{1, t, t^2\}$ for $\mathbb{P}_2$ and the standard basis e for $M_{2\times 2}$ is

$$\begin{bmatrix} [T(1)]_e & [T(t)]_e & [T(t^2)]_e \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}_{4\times 3}$$

We summarize the general method of finding the matrix representation M of T below:

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be any basis for V. If $\mathbf{x} = r_1\mathbf{b}_1 + \cdots + r_n\mathbf{b}_n$, then

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

and since T is linear, we have

$$T(\mathbf{x}) = T\left(r_1\mathbf{b}_1 + \cdots + r_n\mathbf{b}_n\right) = r_1T\left(\mathbf{b}_1\right) + \cdots + r_nT\left(\mathbf{b}_n\right) \tag{1}$$

Since the coordinate mapping from V to $\mathbb{R}^n$ is linear (Theorem 8 in Section 4.4), equation (1) leads to

$$[T(\mathbf{x})]_{\mathcal{C}} = r_1[T\left(\mathbf{b}_1\right)]_{\mathcal{C}} + \cdots + r_n[T\left(\mathbf{b}_n\right)]_{\mathcal{C}} \tag{2}$$

Since $\mathcal{C}$-coordinate vectors are in $\mathbb{R}^m$, the vector equation (2) can be written as a matrix equation, namely

$$[T(\mathbf{x})]_{\mathcal{C}} = M[\mathbf{x}]_{\mathcal{B}} \tag{3}$$

where

$$M = \left[[T\left(\mathbf{b}_1\right)]_{\mathcal{C}} \quad [T\left(\mathbf{b}_2\right)]_{\mathcal{C}} \quad \cdots \quad [T\left(\mathbf{b}_n\right)]_{\mathcal{C}}\right] \tag{4}$$

The matrix M is a matrix representation of T, called the **matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$**.

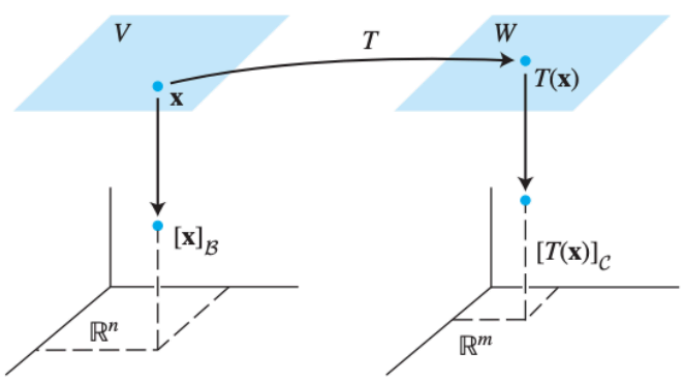


FIGURE 1 A linear transformation from V to W.

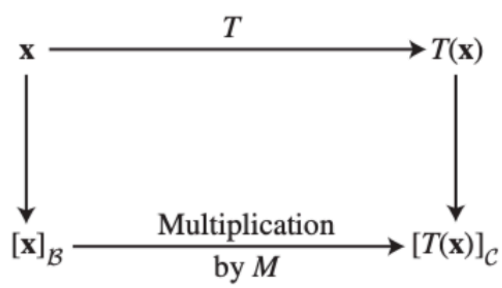


FIGURE 2

Equation (3) says that the action of T on $\mathbf{x}$ can be viewed as left-multiplication by M.

Most of your homework deal with a specal case that $V = W$ and a fixed basis $\mathcal{B}$ for V.

Example 2. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for the vector space V. Let $T : V \to V$ be a linear transformation with the property that $T(\mathbf{b}_1) = 2\mathbf{b}_1 - 6\mathbf{b}_2, T(\mathbf{b}_2) = 3\mathbf{b}_1 + 5\mathbf{b}_2, T(\mathbf{b}_3) = 7\mathbf{b}_2$ Find $[T]_{\mathcal{B}}$, the matrix for T relative to $\mathcal{B}$ (also called the $\mathcal{B}$-matrix for T).

$$\begin{tikzcd}
\mathbf{x} \arrow[r, "T"] \arrow[d] & T(\mathbf{x}) \arrow[d] \\
{[\mathbf{x}]_{\mathcal{B}}} \arrow[r, "\text{Multiplication}", "\text{by } M"'] & {[T(\mathbf{x})]_{\mathcal{B}}}
\end{tikzcd}$$

FIGURE 2

ANS: Note in this case the above discussion reduces to:

Recall in §2.9

We compute $\mathcal{B}$-coordinate vectors of $T(\vec{b}_1)$, $T(\vec{b}_2)$, $T(\vec{b}_3)$

$$[T(\vec{b}_1)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -6 \\ 0 \end{bmatrix} \quad (\text{since } T(\vec{b}_1) = 2\vec{b}_1 - 6\vec{b}_2 + 0\vec{b}_3)$$

similarly, $[T(\vec{b}_2)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$, $[T(\vec{b}_3)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix}$

Thus

$$M = \begin{bmatrix} 2 & 3 & 0 \\ -6 & 5 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

Linear Transformations on $\mathbb{R}^n$

Theorem 8 Diagonal Matrix Representation
Suppose $A = PDP^{-1}$, where D is a diagonal $n \times n$ matrix. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ formed from the columns of P, then D is the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

Example 3. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$. Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that $[T]_{\mathcal{B}}$ is diagonal.

$$A = \begin{bmatrix} 5 & -3 \\ -7 & 1 \end{bmatrix}$$

ANS: Suppose $A = PDP^{-1}$, where D is diagonal.

Then by Thm 8, $[T]_{\mathcal{B}}$ is is D, where $\mathcal{B}$ is the basis for $\mathbb{R}^2$ formed from the columns of P.

covered in §5.3

So we can first diagonalize A to find such matrix P. Then the columns of P are the basis desired.

Recall the steps in §5.3:

- We first find the eigenvalues of A:

$$|A-\lambda I| = \begin{vmatrix} 5-\lambda & -3 \\ -7 & 1-\lambda \end{vmatrix} = (\lambda-1)(\lambda-5)-21 = \lambda^2-6\lambda-16$$

$$= (\lambda-8)(\lambda+2) = 0 \Rightarrow \lambda = -2 \text{ and } \lambda = 8$$

- For $\lambda = -2$, we solve $(A-\lambda I)\vec{x} = \vec{0}$ and get $\vec{v}_1 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ (eigenvector for -2)
- For $\lambda = 8$, we solve $(A-\lambda I)\vec{x} = \vec{0}$, to get $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (eigenvector for 8)

Thus $P = \begin{bmatrix} 3 & -1 \\ 7 & 1 \end{bmatrix}$ and the basis is $\left\{ \begin{bmatrix} 3 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

Change of Basis; Similarity of Matrix Representations

Consider $\mathbb{R}^n$ with standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ and a linear transformation $T : \mathbf{x} \mapsto A\mathbf{x}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Suppose $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$ is another basis for $\mathbb{R}^n$, what is the $\mathcal{B}$-matrix for the transformation $T : \mathbf{x} \mapsto A\mathbf{x}$?

In fact, this is a special case of what we discussed in Figure 2:

$$\begin{tikzcd}
{[\vec{x}]_{\text{standard basis}}} \arrow[r, "\text{multiply by } A"', "T"] \arrow[d, "\text{multiply by } P^{-1}"'] & {[A\vec{x}]_{\text{standard basis}}} \\
{[\vec{x}]_{\mathcal{B}}} \arrow[r, "\text{multiply by } C"] & {[A\vec{x}]_{\mathcal{B}}} \arrow[u, "\text{multiply by } P"']
\end{tikzcd}$$

$A = PCP^{-1}$

or equivalently,

$C = P^{-1}AP$

If we set the matrix $P = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n]$, then the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is

$$C = P^{-1}AP.$$

Remark: The set of all matrices similar to a matrix A coincides with the set of all matrix representations of the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

Example 4. Find the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$, when $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$.

$$A = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}, \mathbf{b}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

ANS: Let $P = [\vec{b}_1 \ \vec{b}_2]$. From the above discussion, we know the $\mathcal{B}$-matrix for the transformation $\vec{x} \mapsto A\vec{x}$ is

$$C = P^{-1}AP.$$

$$P^{-1} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{3+2}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$$

Thus $C = P^{-1}AP$

$$= \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

Then $C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ is the $\mathcal{B}$ matrix for the transformation $\vec{x} \mapsto A\vec{x}$.

Exercise 5. Assume the mapping $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by

$$T\left(a_0 + a_1 t + a_2 t^2\right) = 6a_0 + (3a_0 - 5a_1)t + (3a_1 + 2a_2)t^2$$

is linear. Find the matrix representation of T relative to the basis $\mathcal{B} = \{1, t, t^2\}$.

Solution. We first compute the image of the basis in $\mathcal{B}$ under T:

$T(1) = 6 + 3t, \quad T(t) = -5t + 3t^2$, and $T\left(t^2\right) = 2t^2$.

Thus $[T(1)]_{\mathcal{B}} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$, $[T(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -5 \\ 3 \end{bmatrix}$ and $[T(t^2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

Therefore, $[T]_{\mathcal{B}} = \begin{bmatrix} 6 & 0 & 0 \\ 3 & -5 & 0 \\ 0 & 3 & 2 \end{bmatrix}$